Pricing Forward Start Options in Models based on (time-changed) Lévy Processes

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Abstract

Options depending on the forward skew are very popular. One such option is the forward starting call option - the basic building block of a cliquet option. Widely applied models to account for the forward skew dynamics to price such options include the Heston model, the Heston-Hull-White model and the Bates model. Within these models solutions for options including forward start features are available using (semi) analytical formulas.
Today exponential (subordinated) Levy models being increasingly popular for modelling the asset dynamics. While the simple exponential Levy model imply the same forward volatility surface for all future times the subordinated models do not. Depending on the subordinator the dynamic of the forward volatility surface and therefore stochastic volatility can be modelled. Analytical pricing formulas based on the charcteristic function and Fourier transform methods are available for the class of these models. We extend the applicability of analytical pricing to options including forward start features. To this end we derive the forward characteristic functions which can be used in Fourier transform based methods.
As examples we consider the Variance Gamma model and the NIG model subordinated by a Gamma Ornstein Uhlenbeck process and respectively by an Cox-Ingersoll-Ross process. We check our analytical results by applying Monte Carlo methods. These results can for instance be applied to calibration of the forward volatility surface.

Contents

1 Introduction

The aim of this paper is to derive closed form expressions for the forward characteristic function of time-changed exponential Lévy models. Our modeling assumption are base on Lévy processes. A cadlag stochastic process $(X_t)_t$ with values in $\mathbb{R}^d$ such that $X(0) = 0$ is called a Lévy process if it possesses the following properties:

1 Independent increments:For every increasing sequence times $t_0, \ldots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent

2 Stationary increments: The law of $X_{t+h} - X_t$ does not depend on t.

3 Stochastic continuity: $\forall \epsilon > 0$, $\lim_{\epsilon \to 0} \mathbb{P}\left(|X_{t+\epsilon} - X_t| \geq \epsilon\right) = 0$

We consider financial models of the following type:
Let $(X(t))_t$ be a Lévy process. Then, the evolution of an asset is given by

$$\begin{cases} S(t) &= S(0)\exp(X(t)) \\ S(0) &= s_0 \end{cases} \tag{1.1}$$

If we write the (1.1) in logarithmic form we have

$$\begin{cases} Z(t) &= Z(0) + X(t) \\ Z(0) &= z_0 := \ln(S(0)) \end{cases} \tag{1.2}$$

In this paper the underlying S_t is assumed to follow an exponential time-changed Lévy model of the form

$$S_t = S_0 \frac{\exp((r-d)t)}{\mathbb{E}[\exp(X_{Y_t})]} \exp(X_{Y_t}). \tag{1.3}$$

Here r denotes the risk free rate and d the dividend yield d. We assume that $d = 0$. The underlying S is driven by the Lévy process X time-changed by Y. If $Y(t) = t$ we end up in the class of Lévy models. To make the asset price process into a martingale we can apply the Esscher transform technique or the mean adjustment technique. See Schoutens (2003) for these techniques. We assume that the time-change Y is given as the integral of a positive stochastic process y like

$$Y_t = \int_0^t y(s)\mathrm{d}s. \tag{1.4}$$

This time-change accounts for the concept of stochastic volatility. Several authors use such models to price exotic options, see for example Schoutens, Simons and Tistaert (2003), CARR AND ALL. Recently we have also shown how to use transition probabilities, Kienitz (2008a) and characteristic functions, Kienitz (2008b), to compute stable Greeks for path-dependent options with discontinuous payoff functions using Monte Carlo methods for a processes of the form 1.3. Furthermore, analytic formulas based on the characteristic function and the Fourier transform of the option's payoff function have been introduced by Lewis (2001) to efficiently price options.
The Fourier transform is defined as follows: Let X be a stochastic process than

$$\Phi_X := \mathbb{E}[\exp(iuX)] \tag{1.5}$$

is the Fourier transform or the characteristic function corresponding to X. We have:

$$\mathbb{E}[\exp(izX)] = \int \exp(iuX) d\mathbb{P}_X$$

and if the measure $\mathbb{P}_X$ has a density f with respect to Lebesgue measure than

$$\hat{f}(u) := \Phi_X(u) = \int_S \exp(iux) f(x) dx \tag{1.6}$$

The characteristic exponent given $(X_t)_t$, often used in the sequel, is given by:

$$\psi_X(u) = \log\left(\mathbb{E}[\exp(iuX_1)]\right) \tag{1.7}$$

For an European option with payoff function w and a stochastic asset price process $Z = \log(S_0) + (r-d)T$ we have due to results from Lewis (2001):

$$V(S_0) = \frac{\exp(-rT)}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} \exp(-izZ)\Phi_T(-z)\hat{w}(z)dz \tag{1.8}$$

In equation () $\hat{w}$ denotes the Fourier transform of the option's payoff given by:

$$\hat{w}(z) = \int_{-\infty}^{+\infty} \exp(izx)w(x)dx$$

In the case of a European Call this expression is nothing but

$$\hat{w}_C = -K^{iz+1}/(z^2 - iz)$$

Once the forward characteristic function is available we are able to efficiently compute the forward start values corresponding to this formula.
In this paper we consider exponential Lévy processes with integrated CIR and Gamma Ornstein-Uhlenbeck time-change.
Take a Lévy process $(X_t)_t$ with characteristic exponent ψ_X. Firstly, we consider a Cox-Ingersoll-Ross (CIR) process given by:

$$\left\{ \begin{array}{c} \mathrm{d}y_t = \kappa(\eta - y_t)\mathrm{d}t + \lambda\sqrt{y_t}\mathrm{d}W_t \\ y(0) = 1 \end{array} \right. \tag{1.9}$$

This process is mean reverting, κ is the speed of mean reversion, η is the long-run mean rate and λ controls the volatility of the time-change, see Cox, Ingersoll and Ross (1985) for further details. Besides the CIR process, there was another approach by Carr, Geman, Madan and Yor (2003). They changed time via an Ornstein-Uhlenbeck (OU) process. More explicitly, they took a non-gaussian OU process which has its origin in 2001 (Barndorff-Nielsen and Shephard). It is given by:

$$\left\{ \begin{array}{c} \mathrm{d}y_t = -\lambda y_t \mathrm{d}t + \mathrm{d}z_{\lambda t} \\ y(0) = 1 \end{array} \right. \tag{1.10}$$

In general $z = \{z_t : t \geq 0\}$ can be any Lévy process called the background driving Lévy process (BDLP). Here we assume it is a Compound Poisson process and the marginals of $y(t) \sim \Gamma(a, b)$.
We focus on the Variance Gamma and the Normal Inverse Gaussian model but the results include the classic Black-Scholes-Merton model, the Merton jump diffusion or the Kou model can be considered. Furthermore, we can interpret the Heston stochastic volatility model as a model with stochastic time change. The forward characteristic function for the Heston model is well known.
The characteristic exponent for the Variance Gamma (VG) process is given by:

$$\psi_X^{(\mathrm{VG})}(u) \quad = \quad C \log \frac{GM}{GM + (M-G)\mathrm{i}u + u^2}, \quad C, G, M > 0. \tag{1.11}$$

For the Normal Inverse Gaussian (NIG) process it is given by:

$$\psi_X^{(\mathrm{NIG})}(u) = -\delta(\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2} - \sqrt{\alpha^2 - \beta^2}), \quad \alpha > 0, \alpha < \beta < \alpha, \delta > 0. \tag{1.12}$$

In the sequel we derive the forward characteristic function for the proposed time-changes and an arbitrary Lévy process and an integrated CIR process. Then, we consider the case of a Γ-OU time change already discussed by Kassberger and Schmidt (2006). Sections 2.2.1 and 2.2.2 give the results for the Variance Gamma model and the Normal Inverse Gaussian model. Finally, we compute prices obtained using the forward characteristic function and using Monte Carlo simulation to backtest our analytical formulae.

2 Derivation of Forward Characteristic Function

To fix notation we take t^* to be the forward start time and T the maturity of the option. Since we assume X and Y to be independent we find for the characteristic function of $Z_t \equiv X_{Y_t}$:

$$\phi_{Z_t}(u) \equiv \mathbb{E}[\exp(\mathrm{i}uZ_t)] = \phi_{Y_t}(-\mathrm{i}\psi_X(u)) \tag{2.1}$$

In the above expression ψ_X denotes the characteristic exponent of X which is $\phi_{X_t}(u) = \exp(t\psi_X)$. In order to apply the FFT techniques proposed by Lewis (2001) or that of Carr and Madan (1999), we need to derive the characteristic function of the log forward return of the underlying price process which is given by:

$$s_{t^*,T} = \log\left(\frac{S_T}{S_{t^*}}\right). \tag{2.2}$$

Uaing $\mathbb{E}[\exp(X_{Y_t})] = \phi_{Z_t}(-\mathrm{i})$ we have:

$$\begin{aligned}
\phi_{s_{t^*,T}}(u) &= \mathbb{E}[\exp(\mathrm{i}us_{t^*,T})] && (2.3)\\
&= \exp\left(\mathrm{i}u\left[r(T-t^*) - \log\frac{\phi_{Z_T}(-\mathrm{i})}{\phi_{Z_{t^*}}(-\mathrm{i})}\right]\right)\underbrace{\mathbb{E}[\exp(\mathrm{i}u(Z_T - Z_{t^*}))]}_{=f_1(u)}. && (2.4)
\end{aligned}$$

In the following we derive the forward characteristic function. To this end we take an arbitrary Lévy process X with characteristic exponent ψ_X. For a time-change based on the CIR process ?? we compute the characteristic function corresponding to this process. It is given by:

$$\begin{aligned}
\phi_{y_t}^{(\mathrm{CIR})}(u) &= \left(1 - \frac{1}{2}\mathrm{i}u\frac{\lambda^2}{\kappa}\left(1 - \exp\left(-\kappa t\right)\right)\right)^{-2\kappa\eta/\lambda^2} && (2.5)\\
&\quad \times \exp\left(\frac{\mathrm{i}uy_0\exp(-\kappa t)}{1 - \frac{1}{2}\mathrm{i}u\frac{\lambda^2}{\kappa}(1 - \exp(-\kappa t))}\right). && (2.6)
\end{aligned}$$

We always assume y_0 to be 1. The actual time-change Y which is the integrated CIR process y. It has characteristic function given by:

$$\phi_{Y_t}^{(\mathrm{CIR})}(u) = \frac{\exp(\kappa^2\eta t/\lambda^2)\exp(2y_0\mathrm{i}u/(\kappa + \gamma(u)\coth(\gamma(u)t/2)))}{(\cosh(\gamma(u)t/2) + \kappa\sinh(\gamma(u)t/2)/\gamma(u))^{2\kappa\eta/\lambda^2}} \tag{2.7}$$

Here we used $\gamma(u) = \sqrt{\kappa^2 - 2\lambda^2\mathrm{i}u}$. Now, we calculate $\phi_{s_{t^*,T}}(u)$. To this end we set

$$f_1(u) \equiv \mathbb{E}[\exp(\mathrm{i}u(Z_T - Z_{t^*}))) \tag{2.8}$$

The first step is based on the tower property of conditional expectation. We set:

$$\begin{aligned}
f_1(u) &= \mathbb{E}\left[\mathbb{E}\left[\exp(\mathrm{i}u(X_{t_1} - X_{t_2}))\right]|_{t_1=Y_T, t_2=Y_{t^*}}\right] && (2.9)\\
&= \mathbb{E}\left[\mathbb{E}\left[\exp(\mathrm{i}u(X_{t_1-t_2}))\right]|_{t_1=Y_T, t_2=Y_{t^*}}\right] && (2.10)
\end{aligned}$$

This can be simplified and we get:

$$\begin{aligned}
f_1(u) &= \mathbb{E}\left[\exp((Y_T - Y_{t^*})\psi_X(u))\right] && (2.11)\\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\mathrm{i}(-\mathrm{i})\psi_X(u)\int_{t^*}^T y(s)\mathrm{d}s\right)\middle| y_{t^*}\right]\right] && (2.12)\\
&= \mathbb{E}\left[\phi_{Y_{T-t^*}}^{y_{t^*}}(-\mathrm{i}\psi_X(u))\right] && (2.13)
\end{aligned}$$

where $\phi_{Y_{T-t^*}}^{y_{t^*}}$ is the characteristic function of Y evaluated at $T - t^*$ with initial value y_{t^*}. We assume that the latter expression is of the form

$$\phi_{Y_{T-t^*}}^{y_{t^*}}(u) = f_2(u)\exp(\mathrm{i}f_3(u)y_{t^*}). \tag{2.14}$$

With respect to (2.7), we find:

$$\begin{aligned}
f_2(u) &= \frac{\exp(\kappa^2\eta(T-t^*)/\lambda^2)}{(\cosh(\gamma(u)(T-t^*)/2) + \kappa\sinh(\gamma(u)(T-t^*)/2)/\gamma(u))^{2\kappa\eta/\lambda^2}} && (2.15)\\
f_3(u) &= \frac{2u}{\kappa + \gamma(u)\coth(\gamma(u)(T-t^*)/2)}. && (2.16)
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
f_1(u) &= f_2(-\mathrm{i}\psi_X(u))\mathbb{E}\left[\exp(\mathrm{i}f_3(-\mathrm{i}\psi_X(u))y_{t^*})\right] && (2.17)\\
&= f_2(-\mathrm{i}\psi_X(u))\phi_{y_{t^*}}(f_3(-\mathrm{i}\psi_X(u))) && (2.18)
\end{aligned}$$

with $\phi_{y_{t^*}}$ the characteristic function of y evaluated at t^*, see also (2.5). Finally, in the last three sections we give the explicit expressions of the forward characteristic functions for the CIR-VG, CIR-NIG, Γ-OU-VG, Γ-OU-NIG, VG and NIG model.

2.1 Forward Characteristic Function for Lévy Processes and CIR Time-Change

Set $\gamma(u) = \sqrt{\kappa^2 - 2\lambda^2 \mathrm{i}u}$. Depending on the Lévy process X we have

$$
\begin{aligned}
f_1(u) &= f_2(-\mathrm{i}\psi_X(u)) \left(1 - \frac{1}{2}\mathrm{i}f_3(-\mathrm{i}\psi_X(u))\frac{\lambda^2}{\kappa}\left(1 - \exp\left(-\kappa t^*\right)\right)\right)^{-2\kappa\eta/\lambda^2} && (2.19)\\
&\quad \times \exp\left(\frac{\mathrm{i}f_3(-\mathrm{i}\psi_X(u))y_0\exp(-\kappa t^*)}{1 - \frac{1}{2}\mathrm{i}f_3(-\mathrm{i}\psi_X(u))\frac{\lambda^2}{\kappa}(1 - \exp(-\kappa t^*))}\right) && (2.20)\\
f_2(u) &= \frac{\exp(\kappa^2\eta(T - t^*)/\lambda^2)}{(\cosh(\gamma(u)(T - t^*)/2) + \kappa\sinh(\gamma(u)(T - t^*)/2)/\gamma(u))^{2\kappa\eta/\lambda^2}} && (2.21)\\
f_3(u) &= \frac{2u}{\kappa + \gamma(u)\coth(\gamma(u)(T - t^*)/2)}. && (2.22)
\end{aligned}
$$

The characteristic function for X_{Y_t} at time T evaluated at -i is

$$
\begin{aligned}
\phi^{(\text{X-CIR})}_{Z(T)}(-\mathrm{i}) &= \exp(\kappa^2\eta T/\lambda^2) && (2.23)\\
&\quad \times \exp\left(\frac{2y_0\mathrm{i}(-\mathrm{i})\psi_X(-\mathrm{i})}{\kappa + \gamma((-\mathrm{i})\psi_X(-\mathrm{i}))\coth(\gamma((-\mathrm{i})\psi_X(-\mathrm{i}))T/2)}\right) && (2.24)\\
&\quad \Big/ \left[\cosh\left(\frac{1}{2}\gamma((-\mathrm{i})\psi_X(-\mathrm{i}))T\right)\right.\\
&\quad \left. + \frac{\kappa}{\gamma((-\mathrm{i})\psi_X(-\mathrm{i}))}\sinh\left(\frac{1}{2}\gamma((-\mathrm{i})\psi_X(-\mathrm{i}))T\right)\right]^{2\kappa\eta/\lambda^2}
\end{aligned}
$$

and finally we get:

$$
\phi^{(\text{X-CIR})}_{s_{t^*,T}}(u) = \exp\left(\mathrm{i}u\left[r(T - t^*) - \log\frac{\phi_{Z(T)}(-\mathrm{i})}{\phi_{Z(t^*)}(-\mathrm{i})}\right]\right) f_1(u). \qquad (2.25)
$$

2.2 Forward Characteristic Function for Lévy Processes and Gamma-OU Time-Change

We consider the forward characteristic function of a Lévy process being time-changed with the process from equation (1.10). In this case the characteristic functions for ΓOU process and the integrated process are:

$$
\phi^{(\Gamma-\text{OU})}_{y_t}(u) = \exp\left(\mathrm{i}y_0\exp(-\lambda t)u + a\log\left(\frac{1 - \mathrm{i}/b\exp(-\lambda t)u}{1 - \mathrm{i}/bu}\right)\right) \qquad (2.26)
$$

and

$$
\begin{aligned}
\phi^{(\Gamma-\text{OU})}_{Y_t}(u) &= \exp(\mathrm{i}y_0\lambda^{-1}(1 - \exp(-\lambda t))u && (2.27)\\
&\quad + \frac{\lambda a}{\mathrm{i}u - \lambda b}(b\log\left(\frac{b}{b - iu\lambda^{-1}(1 - \exp(-\lambda t))}\right) - \mathrm{i}ut)).
\end{aligned}
$$

We follow the same procedure as above and obtain:

$$
\begin{aligned}
f_1(u) &= \exp\left(\frac{\mathrm{i}(T - t^*)u\lambda a}{\lambda b - \mathrm{i}u}\right) && (2.28)\\
&\quad \times \exp\left(\frac{ab\lambda}{b\lambda - \mathrm{i}u}\log\left(1 - \frac{\mathrm{i}u}{\lambda b}(1 - \exp(-(T - t^*)\lambda))\right)\right)\\
f_2(u) &= \frac{1}{\lambda}(1 - \exp(-\lambda(T - t^*)))u && (2.29)\\
f_3(u) &= f_1(-\mathrm{i}\Psi_X(u))\exp\left(\mathrm{i}y_0\exp(-\lambda t^*)f_2(-\mathrm{i}\Psi_X(u))\right) && (2.30)\\
&\quad \times \exp\left(a\log\left(\frac{1 - \mathrm{i}/b\exp(-\lambda t^*)f_2(-\mathrm{i}\Psi_X(u))}{1 - \mathrm{i}/bf_2(-\mathrm{i}\Psi_X(u))}\right)\right)
\end{aligned}
$$

Again we use the characteristic exponent and the characteristic function corresponding to Z and we find:

$$\begin{aligned}\phi_{Z(T)}^{(\mathrm{X-\Gamma-OU})}(-\mathrm{i}) &= \exp(\mathrm{i}\psi_X(-\mathrm{i})y_0\lambda^{-1}(1-\exp(-\lambda T)) \\ &\quad + \left[\frac{\lambda a}{\mathrm{i}\psi_X(-\mathrm{i})-\lambda b} \right. \\ &\quad \times \left(b\log\left(\frac{b}{b-\mathrm{i}\psi_X(-\mathrm{i})\lambda^{-1}(1-\exp(-\lambda T))}\right)\right. \\ &\quad \left.\left. -\mathrm{i}\psi_X(-\mathrm{i})T\right)\right]\end{aligned} \tag{2.31}$$

Finally, we get:

$$\phi_{s_{t^*,T}}^{(\mathrm{X-\Gamma-OU})}(u) = \exp\left(\mathrm{i}u\left[r(T-t^*) - \log\frac{\phi_{Z(T)}(-\mathrm{i})}{\phi_{Z(t^*)}(-\mathrm{i})}\right]\right) f_1(u). \tag{2.32}$$

For the sake of completeness we give the forward characteristic functions for the Variance Gamma and the Normal Inverse Gaussian model.

2.2.1 VG Model without Time-Change

For the VG model we have:

$$\phi_{X_1}^{(\mathrm{VG})}(u) = \left(\frac{GM}{GM+(M-G)\mathrm{i}u+u^2}\right)^C \tag{2.33}$$

$$\phi_{X_1}^{(\mathrm{VG})}(-\mathrm{i}) = \left(\frac{GM}{GM+(M-G)-1}\right)^C \tag{2.34}$$

and

$$\begin{aligned}\phi_{s_{t^*,T}}^{(\mathrm{VG})}(u) &= \exp\left(\mathrm{i}u\left[r(T-t^*) - \log\frac{\phi_{X_1}(-\mathrm{i})^T}{\phi_{X_1}(-\mathrm{i})^{t^*}}\right]\right)\phi_{X_1}(u)^{T-t^*} \\ &= \exp\left(\mathrm{i}u(T-t^*)\left[r-\Psi_X^{(\mathrm{VG})}(-\mathrm{i})\right]\right)\phi_{X_{T-t^*}}^{(\mathrm{VG})}(u)\end{aligned} \tag{2.35}$$

2.2.2 NIG Model without Time-Change

For the NIG model we have:

$$\phi_{X_1}^{(\mathrm{NIG})}(u) = \exp\left((-\delta)\left(\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2}-\sqrt{\alpha^2-\beta^2}\right)\right) \tag{2.36}$$

$$\phi_{X_1}^{(\mathrm{NIG})}(-\mathrm{i}) = \exp\left((-\delta)\left(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\right)\right) \tag{2.37}$$

and

$$\phi_{s_{t^*,T}}^{(\mathrm{NIG})}(u) = \exp\left(\mathrm{i}u(T-t^*)\left[r-\Psi_X^{(\mathrm{NIG})}(-\mathrm{i})\right]\right)\phi_{X_{T-t^*}}^{(\mathrm{NIG})}(u). \tag{2.38}$$

3 Results

We illustrate the use of the forward characteristic function by comparing the prices of call options with the prices obtained by applying Monte Carlo simulation. We have chosen to price a call option with maturity of one year starting today, in one, two, three and five years time. Since we base our pricing on the moneyness the process starts at 1. We have chosen $r = 4\%$ and $d = 0\%$. We have chosen to price a European call option maturing in one year from the option start date. The start dates are today, one year, two years and five years. The corresponding prices are labeled as Price1, Price2, Price3 and Price4.
We have applied the standard Fourier transform method together with the derived characteristic functions to price the options. For the Monte Carlo simulation we have used 1.000.000 paths to achieve a small standard error. The Monte Carlo prices can also be seen as a backtest of the derived forward characteristic function.

3.1 VG

We use the parameters $C = 1,2022$, $G = 4,2276$ and $Y = 18,2317$.

Moneyness	Price
0,7	0,33989
0,8	0,25567
0,9	0,17895
1	0,11261
1,1	0,05989
1,2	0,02409
1,3	0,00718

Table 1: Prices with respect to moneyness of a Call option. For the VG process the forward prices are all the same.

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,33945 (0,00021)	0,33975 (0,00021)	0,33953 (0,00021)	0,33992 (0,00021)
0,8	0,25562 (0,00019)	0,25568 (0,00019)	0,25541 (0,00019)	0,25534 (0,00019)
0,9	0,17893 (0,00016)	0,17905 (0,00016)	0,17887 (0,00016)	0,17894 (0,00016)
1	0,11245 (0,00013)	0,11231 (0,00013)	0,11277 (0,00013)	0,11261 (0,00013)
1,1	0,05973 (0,00009)	0,05992 (0,00009)	0,05984 (0,00009)	0,05993 (0,00009)
1,2	0,02425 (0,00006)	0,02398 (0,00006)	0,02401 (0,00006)	0,02409 (0,00006)
1,3	0,00716 (0,00003)	0,00717 (0,00003)	0,00719 (0,00004)	0,00718 (0,00003)

Table 2: Prices computed using the Monte Carlo method

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,0013	0,00041	0,00108	-0,00009
0,8	0,0002	-0,00003	0,00102	0,00128
0,9	0,0001	-0,00055	0,00046	0,00006
1	0,00143	0,00265	-0,00143	0,00003
1,1	0,00275	-0,00046	0,00078	-0,00071
1,2	-0,00675	0,00468	0,00349	-0,00005
1,3	0,00284	0,00137	-0,00128	0,00093

Table 3: The relative errors for comparing the price computed using FFT and the forward characteristic function and Monte Carlo simulation

3.2 NIG

We use the model parameters $\alpha = 8,72609$, $\beta = -6,90168$ and $\delta = 0,169428$.

Moneyness	Price
0,7	0,34241
0,8	0,25761
0,9	0,17946
1	0,11133
1,1	0,05788
1,2	0,0236
1,3	0,00765

Table 4: Prices with respect to moneyness of a Call option. For the VG process the forward prices are all the same.

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,34242 (0,0002)	0,34219 (0,0002)	0,34224 (0,0002)	0,34234 (0,0002)
0,8	0,25746 (0,00018)	0,25758 (0,00018)	0,25775 (0,00018)	0,25743 (0,00018)
0,9	0,17941 (0,00016)	0,17932 (0,00016)	0,17949 (0,00016)	0,17967 (0,00016)
1	0,11157 (0,00013)	0,11155 (0,00013)	0,11128 (0,00013)	0,11157 (0,00013)
1,1	0,05786 (0,00009)	0,05778 (0,00009)	0,05794 (0,00009)	0,05777 (0,00009)
1,2	0,02367 (0,00006)	0,02353 (0,00006)	0,02362 (0,00006)	0,02367 (0,00006)
1,3	0,00767 (0,00004)	0,00764 (0,00004)	0,00761 (0,00004)	0,00765 (0,00004)

Table 5: Prices computed using the Monte Carlo method

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	-0,00003	0,00063	0,0005	0,00021
0,8	0,00059	0,00015	-0,00054	0,00071
0,9	0,0003	0,00079	-0,00018	-0,00114
1	-0,00216	-0,00199	0,00047	-0,00215
1,1	0,00036	0,0019	-0,00094	0,00198
1,2	-0,00271	0,003	-0,00069	-0,003
1,3	-0,00338	0,00059	0,00561	-0,00024

Table 6: The relative errors for comparing the price computed using FFT and the forward characteristic function and Monte Carlo simulation

3.3 VG OU

We use the model parameters $C = 6,47043$, $G = 11,1021$, $M = 33,4128$, $\lambda = 0,939691$, $a = 0,629078$ and $b = 1,46587$.

Moneyness	Price1	Price2	Price3	Price4
0,7	0,33639	0,33169	0,32527	0,32026
0,8	0,25081	0,24659	0,24021	0,23509
0,9	0,1738	0,16869	0,16218	0,15685
1	0,1092	0,10131	0,09421	0,08848
1,1	0,06025	0,04873	0,0405	0,034
1,2	0,02807	0,0155	0,00765	0,00326
1,3	0,01066	0,0028	0,00087	0,00039

Table 7: Prices with respect to moneyness of a Call option. For the VG process the forward prices are all the same.

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,33674 (0,00021)	0,33258 (0,00019)	0,32599 (0,00018)	0,32071 (0,00017)
0,8	0,25122 (0,00019)	0,24721 (0,00017)	0,24082 (0,00016)	0,23584 (0,00015)
0,9	0,17428 (0,00017)	0,16917 (0,00014)	0,16277 (0,00013)	0,15727 (0,00012)
1	0,10959 (0,00014)	0,10183 (0,00011)	0,09467 (0,0001)	0,08873 (0,00009)
1,1	0,06034 (0,0001)	0,04897 (0,00008)	0,04068 (0,00006)	0,03416 (0,00005)
1,2	0,02812 (0,00007)	0,01564 (0,00004)	0,0076 (0,00003)	0,00324 (0,00002)
1,3	0,01072 (0,00004)	0,00276 (0,00002)	0,00086 (0,00001)	0,0004 (0,00001)

Table 8: Prices computed using the Monte Carlo method

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	-0,00104	-0,00266	-0,00221	-0,0014
0,8	-0,00163	-0,00254	-0,00255	-0,00315
0,9	-0,00273	-0,00286	-0,00362	-0,00266
1	-0,00348	-0,0051	-0,00484	-0,00274
1,1	-0,00158	-0,00488	-0,00433	-0,00454
1,2	-0,00189	-0,00919	0,00665	0,00647
1,3	-0,0052	0,01282	0,00839	-0,02205

Table 9: The relative errors for comparing the price computed using FFT and the forward characteristic function and Monte Carlo simulation

3.4 VG CIR

We use the model parameters $C = 58,1151$, $G = 50,4954$, $M = 69,3685$, $\kappa = 1,23294$, $\eta = 0,64976$ and $\lambda = 1,43335$.

Moneyness	Price1	Price2	Price3	Price4
0,7	0,3354	0,32847	0,32468	0,32474
0,8	0,24966	0,24388	0,23952	0,23923
0,9	0,17242	0,1667	0,16171	0,16101
1	0,10734	0,10017	0,09479	0,09377
1,1	0,05783	0,04852	0,04358	0,0425
1,2	0,02566	0,01625	0,0132	0,01253
1,3	0,00909	0,0036	0,00268	0,00249

Table 10: Prices with respect to moneyness of a Call option. For the VG process the forward prices are all the same.

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,33548 (0,0002)	0,32825 (0,00019)	0,3247 (0,00018)	0,32461 (0,00018)
0,8	0,24958 (0,00019)	0,24408 (0,00017)	0,23947 (0,00017)	0,23913 (0,00016)
0,9	0,17253 (0,00016)	0,16654 (0,00015)	0,16192 (0,00014)	0,16119 (0,00014)
1	0,1073 (0,00013)	0,10015 (0,00011)	0,09494 (0,00011)	0,09387 (0,00011)
1,1	0,0579 (0,0001)	0,04852 (0,00008)	0,04354 (0,00007)	0,04244 (0,00007)
1,2	0,02568 (0,00006)	0,01628 (0,00004)	0,01316 (0,00004)	0,0125 (0,00004)
1,3	0,0091 (0,00004)	0,0036 (0,00002)	0,00268 (0,00002)	0,00252 (0,00002)

Table 11: Prices computed using the Monte Carlo method

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	-0,00023	0,00065	-0,00006	0,0004
0,8	0,00033	-0,00084	0,00025	0,00039
0,9	-0,00063	0,00096	-0,00131	-0,00109
1	0,00033	0,00027	-0,0016	-0,00109
1,1	-0,00108	-0,00004	0,00082	0,00154
1,2	-0,00086	-0,00167	0,00319	0,00238
1,3	-0,00169	0,00229	0,00043	-0,0085

Table 12: The relative errors for comparing the price computed using FFT and the forward characteristic function and Monte Carlo simulation

3.5 NIG OU

We use the model parameters $\alpha = 15,9532$, $\beta = -10,6732$, $\delta = 0,412095$, $\lambda = 0,855302$, $a = 0,68104$ and $b = 1,52999$.

Moneyness	Price1	Price2	Price3	Price4
0,7	0,33741	0,33275	0,32622	0,32034
0,8	0,25214	0,2477	0,24113	0,23506
0,9	0,17507	0,16969	0,1629	0,15653
1	0,10992	0,10196	0,09456	0,08773
1,1	0,06027	0,04899	0,04058	0,03312
1,2	0,0278	0,01601	0,00864	0,00366
1,3	0,01053	0,00328	0,00108	0,00045

Table 13: Prices with respect to moneyness of a Call option. For the VG process the forward prices are all the same.

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,33777 (0,00021)	0,33336 (0,00019)	0,32661 (0,00018)	0,32056 (0,00017)
0,8	0,25238 (0,00019)	0,24785 (0,00017)	0,24175 (0,00016)	0,23545 (0,00015)
0,9	0,17531 (0,00017)	0,17005 (0,00014)	0,16344 (0,00013)	0,15693 (0,00012)
1	0,11028 (0,00014)	0,10233 (0,00011)	0,09483 (0,0001)	0,08787 (0,00009)
1,1	0,0604 (0,0001)	0,04925 (0,00008)	0,04083 (0,00006)	0,03324 (0,00005)
1,2	0,02783 (0,00007)	0,01606 (0,00004)	0,00864 (0,00003)	0,00363 (0,00002)
1,3	0,01051 (0,00004)	0,00328 (0,00002)	0,00109 (0,00001)	0,00044 (0,00001)

Table 14: Prices computed using the Monte Carlo method

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	-0,00105	-0,00183	-0,00121	-0,0007
0,8	-0,00093	-0,00062	-0,00259	-0,00166
0,9	-0,00134	-0,00215	-0,00334	-0,00258
1	-0,00327	-0,00368	-0,00282	-0,00156
1,1	-0,00204	-0,00539	-0,00625	-0,00362
1,2	-0,00118	-0,00325	-0,00031	0,00766
1,3	0,00187	-0,00009	-0,0064	0,01716

Table 15: The relative errors for comparing the price computed using FFT and the forward characteristic function and Monte Carlo simulation

3.6 NIG CIR

We use the model parameters $\alpha = 538,311$, $\beta = -9,58059$, $\delta = 17,9022$, $\kappa = 1,32756$, $\eta = 0,656748$ and $\lambda = 1,50423$.

Moneyness	Price1	Price2	Price3	Price4
0,7	0,33556	0,32846	0,32518	0,32532
0,8	0,24984	0,24382	0,24001	0,23986
0,9	0,17256	0,16657	0,16216	0,16169
1	0,10737	0,09999	0,09523	0,09449
1,1	0,05778	0,04844	0,0441	0,0433
1,2	0,02567	0,01651	0,01386	0,01336
1,3	0,00917	0,00381	0,00295	0,0028

Table 16: Prices with respect to moneyness of a Call option. For the VG process the forward prices are all the same.

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,33603 (0,0002)	0,32839 (0,00019)	0,3249 (0,00018)	0,32524 (0,00018)
0,8	0,25001 (0,00019)	0,24355 (0,00017)	0,24002 (0,00017)	0,23998 (0,00016)
0,9	0,17262 (0,00016)	0,16683 (0,00015)	0,16232 (0,00014)	0,16148 (0,00014)
1	0,10753 (0,00013)	0,09997 (0,00011)	0,09536 (0,00011)	0,09443 (0,00011)
1,1	0,05772 (0,0001)	0,04845 (0,00008)	0,04412 (0,00007)	0,04324 (0,00007)
1,2	0,02556 (0,00006)	0,01642 (0,00005)	0,01386 (0,00004)	0,01336 (0,00004)
1,3	0,00918 (0,00004)	0,00378 (0,00002)	0,00295 (0,00002)	0,00282 (0,00002)

Table 17: Prices computed using the Monte Carlo method

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	-0,00138	0,00022	0,00084	0,00026
0,8	-0,00066	0,00113	-0,00004	-0,0005
0,9	-0,00036	-0,00156	-0,00098	0,00134
1	-0,00148	0,0002	-0,00135	0,00058
1,1	0,00113	-0,00028	-0,0005	0,00144
1,2	0,004	0,00536	-0,00056	-0,00028
1,3	-0,00059	0,00685	-0,00048	-0,00803

Table 18: The relative errors for comparing the price computed using FFT and the forward characteristic function and Monte Carlo simulation

3.7 Merton Jump

We use the model parameters $\sigma = 0,152861$, $\alpha_j = -0,88229$, $\sigma_j = 0,000271941$ and $\lambda = 0,0813494$.

Moneyness	Price
0,7	0,34695
0,8	0,25929
0,9	0,17634
1	0,10603
1,1	0,05551
1,2	0,0253
1,3	0,01015

Table 19: Prices with respect to moneyness of a Call option. For the VG process the forward prices are all the same.

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	0,34723 (0,00019)	0,34711 (0,00019)	0,3475 (0,00019)	0,34692 (0,00019)
0,8	0,25906 (0,00018)	0,25915 (0,00018)	0,25938 (0,00018)	0,2593 (0,00018)
0,9	0,17641 (0,00016)	0,17617 (0,00016)	0,17627 (0,00016)	0,17639 (0,00016)
1	0,10595 (0,00013)	0,10604 (0,00013)	0,10599 (0,00013)	0,10608 (0,00013)
1,1	0,05554 (0,0001)	0,05568 (0,0001)	0,0555 (0,0001)	0,05546 (0,0001)
1,2	0,0254 (0,00007)	0,0253 (0,00007)	0,02528 (0,00007)	0,02532 (0,00007)
1,3	0,01016 (0,00004)	0,01012 (0,00004)	0,01012 (0,00004)	0,01014 (0,00004)

Table 20: Prices computed using the Monte Carlo method

Moneyness	Price 1	Price 2	Price 3	Price 5
0,7	-0,0008	-0,00044	-0,00158	0,00008
0,8	0,0009	0,00057	-0,00032	-0,00002
0,9	-0,00039	0,00096	0,00037	-0,0003
1	0,00083	-0,0001	0,00043	-0,00044
1,1	-0,00055	-0,00306	0,00011	0,00093
1,2	-0,00412	0,00005	0,00073	-0,00085
1,3	-0,00152	0,00294	0,00239	0,00071

Table 21: The relative errors for comparing the price computed using FFT and the forward characteristic function and Monte Carlo simulation

4 Conclusion

The derived forward characteristic functions can efficiently be applied to pricing problems involving forward start options using the Fast Fourier Transform (FFT) as suggested by Lewis (2001). To this end the forward implied volatility surface can be considered for risk management and option calibration. In a future paper, Beyer and Kienitz (2009), we show how the choice of the time-change process affects the shape of the implied forward volatility surface.
An in-depth study of the forward implied volatilities can be done using the described techniques. This is interesting for calibration, risk management and pricing in the framework of time-changed Lévy processes.

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