

A Note on Monte Carlo Greeks using the Characteristic Function

Jörg Kienitz, Deutsche Postbank AG

email: joerg.kienitz@postbank.de

Version; 0.1

November 28, 2008

Abstract

We consider the derivation of generic Monte Carlo estimators for Greeks for (path-dependent) options with discontinuous payoffs in the case where only the characteristic function is known. In Kienitz (2008) we have shown how to derive such Greeks for a wide range of models under the assumption that the transition probability is known in closed form. Unfortunately, this is not always the case. For example when considering exponential Lévy models with stochastic volatility such as the Variance Gamma model with a Gamma Ornstein-Uhlenbeck or CIR stochastic clock.

The characteristic function in this case the density is only given through its characteristic function. We give an algorithm to compute the probability density from the characteristic function and show that computing the transition density in this way gives the same results as in Kienitz (2008) but works for very general models.

In this paper we focus on the Variance Gamma model and the same model with a Gamma Ornstein-Uhlenbeck stochastic clock.

Since the methods are very general we can cope with other complex models like the Normal Inverse Gaussian model, considering other types of stochastic clocks or other classes of models where the characteristic function is known.

Contents

1	Introduction and Objectives	2
2	The Modeling Setup	3
2.1	Lévy Processes	4
2.2	The Variance Gamma Model (VG)	4
2.2.1	Monte Carlo Simulation	5
2.3	The Variance Gamma Model with Gamma Ornstein-Uhlenbeck Clock (VGOU)	6
2.3.1	Simulation	7

3	The Probability Density and the Characteristic Function	8
3.1	Error Bounds	9
3.2	The Algorithm	9
3.3	Source of Error	9
3.4	Applying the Method to the VG model	10
3.5	Applying the Method to the VGOU model	12
3.5.1	Varying a	13
3.5.2	Varying b	14
4	Estimating Sensitivities using Proxy Methods	15
4.0.3	The Proxy Method and Algorithm	15
5	Numerical Examples	17
5.1	The VG model	17
5.1.1	Digital Option	17
5.1.2	Knock-Out Options	19
5.2	The VGOU model	22
5.2.1	Digital Option	22
5.2.2	Knock-Out Option	23
6	Summary and Conclusion	26

1 Introduction and Objectives

Financial Institutions apply advanced models to risk manage their trading books. The methodology has gone far beyond the initial Black-Scholes-Merton model. Furthermore, the options involved have complex payoff structures involving path-dependency and discontinuities along the path.

Thus, prices of complex exotic structures can only be approximated using Monte Carlo methods. For an account on stochastic processes in finance and Monte Carlo methods see Glasserman (2004), Jäckel (2002), Kienitz (2007a), Kienitz (2007b) and Kienitz (2009). For a detailed introduction to the implementation of such methods we refer to Duffy and Kienitz (2009).

The payoff functions under consideration may also have discontinuities. The calculation of price sensitivities for such exotic options is one of the central modeling and computational problems.

The Greeks are computed as the derivative of the price due to several model parameters. If this expectation is estimated using Monte Carlo simulation the derivatives can be approximated using finite differences, pathwise estimation or likelihood ratio methods. Derivative based methods are not well suited for covering functions with discontinuities. Applying likelihood ratio methods is grounded on differentiation on the path space. The calculus on the path space, commonly known as Malliavin calculus involves heavy mathematical machinery. Here, the aim is to establish a chain rule as well as an integration by parts formula on the path space. For a certain class of jump diffusion processes this

has been done by León (2002) and has been used and extended for financial modeling by Davis (2004). In the latter paper Davis and Johansson consider jump diffusions where the jump amplitude is deterministic. In this case they obtain likelihood ratio weights for several jump diffusion models.

However, for Lévy processes there is no chain rule for the Malliavin derivative and the above arguments cannot be applied. El-Khatib (2004) considered a market driven by jumps alone. They performed Malliavin calculus on the Poisson space and are able to consider processes having Poisson jump times with random jump sizes but imposing a regularity condition on the payoff. Especially, they cannot handle standard European options nor options with discontinuous payoffs.

We extend our numerical method introduced in Kienitz (2008) to compute Greeks for models where the transition probability is not given directly but only through its characteristic function. Our method covers our previous work but also models with stochastic volatility such as exponential Lévy models with stochastic clocks. For instance we test our implementation on the well known Variance Gamma model and the Variance Gamma model with Gamma Ornstein-Uhlenbeck clock. The latter is a stochastic volatility variant of the original variance gamma model. Recently, Glasserman and Liu (2008) and Fang and Osterlee (2008) have used numerical inversion applied to financial problems. Liu and Glasserman use the characteristic function and its derivative to extend the likelihood ratio method for sensitivity analysis while Fang and Osterlee use their fourier cosine series method to price European options. Our paper is organised as follows:

In section 2 we introduce the modeling framework. We recall the definition of a Lévy process and state the main examples we are examining. To finally apply the models to certain financial problems we review the simulation methods and some of the main characteristics of the models. The main tool we use is the numerical inversion of the Fourier transform. The relations and the algorithm we use are introduced in section 3. We apply the methods to certain parameter sets for each model we consider. In section 4 we review the methods to the computation of hedge sensitivities of digital and knock-out options and finally in 5 give numerical results for applying the proposed algorithms to our benchmark examples.

Together with research presented in Beyer and Kienitz (2009a) and Beyer and Kienitz (2009b) we can use these methods to set up a risk management procedure using exponential Lévy models.

2 The Modeling Setup

In this section we discuss the modeling setup. We give the definition of a Lévy process and introduce the financial models we consider as examples, namely the Variance Gamma model (VG) and the Variance Gamma model with Gamma Ornstein-Uhlenbeck stochastic clock (VGOU).

2.1 Lévy Processes

Let us start by describing the modeling framework. To this end we recall the definition of a Lévy process, taken from Cont and Tankov (2004).

A cadlag stochastic process $(X(t))_t$ with values in \mathbb{R}^d such that $X(0) = 0$ is called a Lévy process if it possesses the following properties:

- 1 Independent increments: For every increasing sequence times t_0, \dots, t_n , the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent
- 2 Stationary increments: The law of $L_{t+h} - L_t$ does not depend on t .
- 3 Stochastic continuity: $\forall \epsilon > 0, \lim_{\epsilon \rightarrow 0} \mathbb{P}(|L_{t+\epsilon} - L_t| \geq \epsilon) = 0$

We consider financial models of the following type:

Let $(L(t))_t$ be a Lévy process. Then, the evolution of an asset is given by

$$(1) \quad \begin{cases} S(t) &= S(0) \exp(L(t)) \\ S(0) &= s_0 \end{cases}$$

If we write the (1) in logarithmic form we have

$$(2) \quad \begin{cases} X(t) &= X(0) + L(t) \\ X(0) &= x_0 := \ln(S(0)) \end{cases}$$

There have been some research on the derivation of Greeks for Lévy processes including El-Khatib (2004), Chen (2006), Davis (2004) and León (2002).

2.2 The Variance Gamma Model (VG)

We consider the Variance Gamma process X_t because there is a strong ongoing interest in applying this model in finance. The VG process has Lévy density

$$(3) \quad l_{VG}(x) = \frac{\exp\left(\frac{\mu x}{\sigma^2}\right)}{\nu |x|} \exp\left(-\frac{|x|}{\sigma} \sqrt{\frac{2}{\nu} + \frac{\mu^2}{\sigma^2}}\right),$$

and characteristic function

$$(4) \quad \mathbb{E} [e^{-iuX_t}] = \left(\frac{1}{1 - i\mu\nu u - \frac{\sigma^2 \nu u^2}{2}} \right)^{t/\nu}.$$

The parameters describe the mean and sign of skewness, μ , the kurtosis ν and σ is a volatility parameter.

The density f_{VG} given the time increment Δ starting at x_t is given by:

$$(5) \quad f_{VG}(x, t + \Delta; x_t, t) = \frac{2 \exp\left(\frac{\mu z(x)}{\sigma^2}\right)}{\nu \Delta / \nu \sqrt{2\pi\sigma} \Gamma(\Delta/\nu)} \left(\frac{z(x)^2}{\frac{2\sigma^2}{\nu} + \mu^2} \right)^{\frac{\Delta}{2\nu} - \frac{1}{4}} \cdot K_{\Delta/\nu - 1/2} \left(\frac{1}{\sigma^2} \sqrt{z(x)^2 \left(\frac{2\sigma^2}{\nu} + \mu^2 \right)} \right)$$

where $z(x) = x - x_t - (r + \omega)\Delta_t$ and $K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu-1} \exp(-\frac{z}{2}(y + y^{-1})) dy$ denotes the modified Bessel function of the second kind. There is yet another representation based on Madan (1998). There, the VG process is modeled by the difference of two independent Gamma processes U_t and D_t , such that $U \sim \Gamma(C, G)$ and $D \sim \Gamma(C, M)$ and

$$(6) \quad X(t) = U(t) - D(t)$$

Both approaches are equivalent and we can use

$$\begin{aligned} C &= \frac{1}{\nu} \\ G &= (\sqrt{\mu^2\nu^2/4 + \sigma^2\nu/2} - \mu\nu/2)^{-1} \\ M &= (\sqrt{\mu^2\nu^2/4 + \sigma^2\nu/2} + \mu\nu/2)^{-1} \end{aligned}$$

to convert the parameters from one setting to the other.

2.2.1 Monte Carlo Simulation

There are two popular methods to simulate from a VG process. The first approach is based on subordinating a standard Brownian motion and the second is to use the representation of two independent Gamma processes.

A subordinator is an increasing process. Let $(M_t)_t$ be the subordinator which is independent of $(W_t)_t$. The process

$$(7) \quad Y(t) := W(M(t))$$

is called a process obtained by subordinating the Brownian motion $W(t)$.

The subordinator in our case is a Gamma process. Subordinating a Brownian motion is straight forward if simulating the increments of the subordinator is straight forward. We state the algorithm to obtain the next step in a sequential Monte Carlo procedure. This algorithm can be found in standard literature like Duffy and Kienitz (2009) or Glasserman (2004).

1. Simulate $Z \sim \mathcal{N}(0, 1)$
2. Simulate $G \sim I(\dots)$
3. Set $X(t + \Delta) = X(t) + r\Delta + \mu\nu G + \sigma\sqrt{\nu G}Z$

The random variable $I()$ is the distribution of the subordinator. It is common practice to adjust the drift of the process such that $S(t)\exp(-rt)$ becomes a martingale. We denote the drift correction at each time point by $\bar{\omega}(t)$.

For the Variance Gamma process the second step is:

Simulate $G \sim \nu\Gamma(\Delta/\nu, 1)$

If we wish to assure that this asset price process is a martingale we have to add the drift correction $\bar{\omega} = \frac{1}{\nu} \log \left(1 - \mu\nu - \frac{\nu\sigma^2}{2} \right)$. The discretisation is given by:

$$(8) \quad X(t + \Delta) = X(t) + (r + \bar{\omega})\Delta + \mu G + \sigma\sqrt{G}Z$$

Sampling using 6 is straight forward because the increments are those of an ordinary Γ process.

2.3 The Variance Gamma Model with Gamma Ornstein-Uhlenbeck Clock (VGOU)

Denoting again the VG process by X_t and consider a process of the form $X(Y_t)$ with Y_t independent of X . The process Y_t represents the time change or stochastic clock.

For the case of a Gamma Ornstein-Uhlenbeck process we take the solution $\{y_t, 0 \leq t \leq T\}$ of the stochastic differential equation:

$$(9) \quad \begin{cases} dy(t) = -\lambda y(t)dt + dz(\lambda t) \\ y(0) = y_0 \end{cases}$$

where z_t is a compound Poisson process. For the stochastic clock we take the process:

$$(10) \quad Y_t = \left\{ \int_0^t y_s ds, 0 \leq t \leq T \right\}$$

The asset price dynamics for the VGOU model is now given by:

$$(11) \quad S(t) = \frac{\exp(rt)}{\mathbb{E}[\exp(X(Y(t)))|y_0]} \exp(X(Y(t)))$$

For the characteristic function of the logarithm of the stock price S_t we have:

$$(12) \quad \exp(iurt + \log(S(0))) \frac{\Phi(-i\psi_{VG}(u); t, y_0)}{\Phi(-i\psi_{VG}(-i); t, y_0)^{iu}}$$

with *characteristic exponent*

$$(13) \quad \psi_{VG}(u) = \log(\mathbb{E}[\exp(iuX(1))])$$

The latter function, (13), is called the characteristic exponent.

$$(14) \quad \Phi_{VGOU}(u) = \exp(iu(rT + \log(S(0)) - \log(\Phi(-i)))) f_3(u)$$

For the model under consideration this function is given by:

$$\begin{aligned} \Phi(-i) &= \exp\left(i\psi_{VG}(-i)y_0\lambda^{-1}f_\lambda\right) \\ &+ \frac{\lambda}{i\psi_{VG}(-i) - \lambda b} \left(b \log\left(\frac{b}{b - i\psi_{VG}(-i)\lambda^{-1}f_\lambda}\right) - i\psi_{VG}(-i)T \right) \end{aligned}$$

We used the notation

$$\begin{aligned}
\psi_{VG}(u) &= -C \log \left(\frac{GM}{GM + (M - G)iu + u^2} \right) \\
f_1(u) &= \exp \left(\frac{iT\psi_{VG}(u)\lambda a}{\lambda b - i\psi_{VG}(u)} + \frac{ab\lambda}{b\lambda - i\psi_{VG}(u)} \log \left(1 - \frac{i\psi_{VG}(u)}{\lambda b} f_\lambda \right) \right) \\
f_2(u) &= \frac{1}{\lambda} f_\lambda \psi_{VG}(u) \\
f_3(u) &= f_1(u) \exp(iy_0 f_2(u)) \\
f_\lambda &= (1 - \exp(-\lambda T))
\end{aligned}$$

For the VGOU model the martingale correction can be computed by using

$$(15) \quad \bar{\omega} = \exp((r - q)t / \varphi(-i\psi_{VG}(-i); t, 1))$$

and

$$(16) \quad \varphi(-i\psi_X^{VG}(-i); t, 1) = TTTT$$

2.3.1 Simulation

To simulate the stochastic clock we first take a discretisation $\mathcal{T} = \{t_0 = 0, t_1, \dots, t_N = T\}$ of $[0, T]$ and consider the process y_t given by:

$$(17) \quad y_{t_n} = (1 - \lambda\Delta t)y_{t_{n-1}} + \sum_{k=N_{t_{n-1}}+1}^{N_{t_n}} x_k \exp(-\lambda\Delta t u_k)$$

with random variates given by:

$$u_k \sim U(0, 1); \quad x_k \sim \log(\bar{u}_k)/b, \quad \bar{u}_k \sim U(0, 1)$$

and the process N_t being a Poisson process,

$$N_t \sim \text{Poisson}(a\lambda t)$$

In practice we have two numbers $NTimes$ and $NIntTimes$ which determine the discretisation. On the one hand side we have the number of times where the process is monitored, $NTimes$, for example this could be monthly or quarterly and on the other hand side we need the time points which are used to compute the business time. This number of points is determined by $NIntTimes$.

Then, we compute the integrated process Y_t by setting

$$(18) \quad Y_t = \int_0^t y_s ds$$

Finally, for

$$t_i \in \mathcal{T}$$

set

$$X_{t_i} = X_{Y_{t_i}}$$

Therefore, to sample from the VGOU process we proceed as follows:

- (i) Simulate the process y_t using (17)
- (ii) Compute the process Y_t using the process y
- (iii) Simulate the process X_t
- (iv) Simulate the time changed process $X(Y_t)$

3 The Probability Density and the Characteristic Function

This section briefly reviews some facts on computing the probability density using the corresponding characteristic function. We sketch an efficient algorithm which we use for computing sensitivities applying Monte Carlo simulation. The characteristic function and the probability density of random variable Z are related by:

$$(19) \quad \hat{f}(u) = \int_S \exp(iux) f(x) dx$$

$$(20) \quad f(x) = \frac{1}{2\pi} \int_S \exp(-iux) \hat{f}(u) f u$$

we consider f on the discrete set

$$u_k = \pi/A(k - N/2), k = 0, \dots, N - 1$$

which leads to:

$$\hat{f}(u_k) = \int_S \exp(iu_k x) f(x) dx \approx \Delta x \sum_{j=0}^{N-1} \exp(iu_k x_j) f(x_j) \quad k = 0, \dots, N - 1$$

Using conjugate symmetry we find:

$$\begin{aligned} & \exp(-iu_k x_j) \phi_t(u_k) + \exp(-iu_{N-k} x_j) \phi_t(u_{N-k}) \\ &= \exp(-iu_k x_j) \phi_t(u_k) + \exp(iu_k x_j) \overline{\phi_t(u_k)} \\ &= \exp(-iu_k x_j) \phi_t(u_k) + \overline{\exp(-iu_k x_j) \phi_t(u_k)} \\ &= 2\mathcal{R}(\exp(-iu_k x_j) \phi_t(u_k)) \end{aligned}$$

The latter equation is used to implement the inversion technique we apply for the proxy simulation if only the characteristic function is given.

In Fang and Osterlee (2008) it is suggested to use a Fourier-Cosine series to numerically invert the characteristic function. They use their method to price European options. Corresponding error bounds for their methods are also derived.

3.1 Error Bounds

For the error bound we use Theorem 7 from Hughett (1998). To this end let d denote a probability distribution and ϕ the corresponding characteristic function. If there exist positive numbers $B, C, \alpha > 1$ and some constant $\beta > 1/2$ with

$$|d(x)| \leq B|x|^{-\alpha}, \quad \text{for all } |x| > A,$$

and

$$|\phi(u)| \leq C|u/(2\pi)|^{-\beta}, \quad \text{for all } |u| > \frac{\pi N}{2A}.$$

Then,

$$\frac{1}{\sqrt{2A}} \|d - d_s\| \leq \frac{B}{A^\alpha} \frac{2\alpha - 1}{\alpha - 1}$$

and

$$\frac{1}{\sqrt{2A}} \|\|\| \leq \frac{C(2A)^{\beta-1/2}}{\sqrt{\beta-1/2}} \left(\frac{2}{N}\right)^{\beta-1/2}$$

where we denote by d_s the periodified density d as in Hughett (1998).

3.2 The Algorithm

Consider the interval $S = [-A, A]$. We assume that the interval is chosen such that $d(-A) = d(A)$. Let M be a positive integer. To determine the value of the probability density at x we proceed by:

- Set $N = 2^M$
- $u_j = j\pi(1/A - N/(2A)), \quad j = 0, \dots, N - 1$
- Set $\delta_u = \pi/A$
- return $\sum_{j=0}^{N-1} 2(\mathcal{R}(\exp(-iu_j x) f(u_j))) \Delta_u / (4\pi)$;

We denote by $\mathcal{R}(\cdot)$ the real part of a complex number.

We remark that the interval S need not be symmetric. We have assumed symmetry just for convenience.

Fang and Osterlee have reported in Fang and Osterlee (2008) that their methods lead to stable and robust results and are specially suited to reduce computational effort.

3.3 Source of Error

The choice of the interval S may lead to unreasonable behaviour if A is too small. Since the algorithm produces a periodic function especially using asymmetric distribution functions may suffer from an inappropriate value of A . We identify two errors affecting the approximation:

- Truncation Error (choosing A)

- Discretisation Error (choosing M)

We could ask ourselves on an appropriate constant A . For example we have applied the following choices in practical experiments:

The first choice is based on the observation that

$$\mathbb{E}[X] = -i \frac{d}{du} \Phi_X(u) \Big|_{u=0}$$

$$\mathbb{V}[X] = -i \frac{d^2}{du^2} \Phi_X(u) \Big|_{u=0}$$

Then, computing the first and second order finite differences

$$\begin{aligned} \bar{\mu} &= -i \frac{\Phi_X(\epsilon) - \overline{\Phi_X(\epsilon)}}{2\epsilon} \\ \bar{\sigma} &= \sqrt{\frac{\Phi_X(\epsilon) - 2 + \overline{\Phi_X(\epsilon)}}{\epsilon^2} - \bar{\mu}^2} \end{aligned}$$

we can take

$$[-n\bar{\sigma}, n\bar{\sigma}]$$

to include all approximately the n standard deviations for the numerical inversion.

Another choice is motivated by the fact that we are dealing with probability densities. Thus, the integral of the density is equal to 1. We may compute the integral starting from the expected value to the left and to the right and stop if we have reached a certain boundary, i.e. we take $A = \mu - \tilde{A}$ such that

$$\int_{\mu-\tilde{A}}^{\mu+\tilde{A}} f(x) dx \approx 1.$$

3.4 Applying the Method to the VG model

As a benchmark model we consider the VG model. To this end we consider the following parameter sets for our numerical tests:

Set	C	G	M
Set 1	6.161	9.6443	16.026
Set 2	8.88	24.95	48.19
Set 3	6.47	11.10	33.41

Table 1: The parameter sets for numerical experiments

For each of the parameter sets we plot the density which is given in closed form for the VG model and compare it to our approximation using the characteristic function. Set1 is motivated by the choice of parameters by Schoutens

et al. (2004). The sets Set2 and Set3 are obtained by calibrating the model to index data for the STOXX50 and DAX indices.

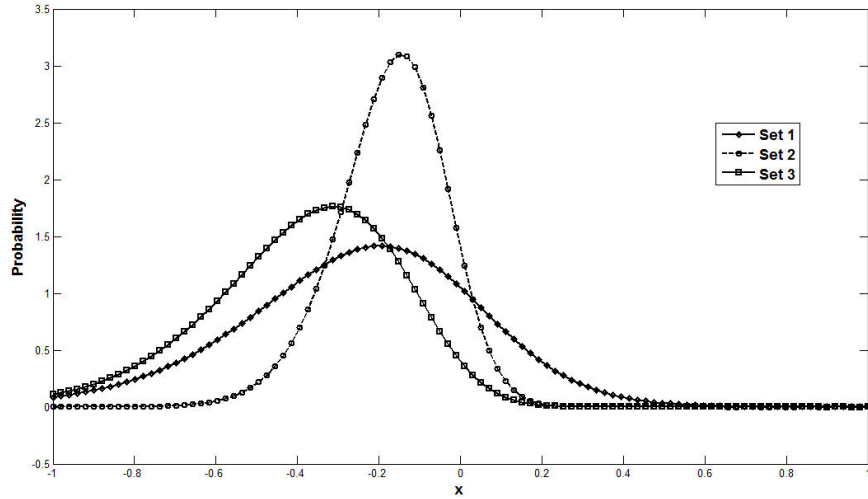


Figure 1: Difference between the probability distributions obtained by using the characteristic function and the closed form solution. The lines indicate the values computed using the closed form solution and the indicated points show the values obtained using the characteristic function

We consider the approximation for different times. Figures 2 and 3 illustrate the probability distribution computed using the closed form solution (upper left), the method using the characteristic function (upper right), the absolute error (lower left) and both distributions (lower right). We see that the approximation error is lesser for longer time to maturity T .

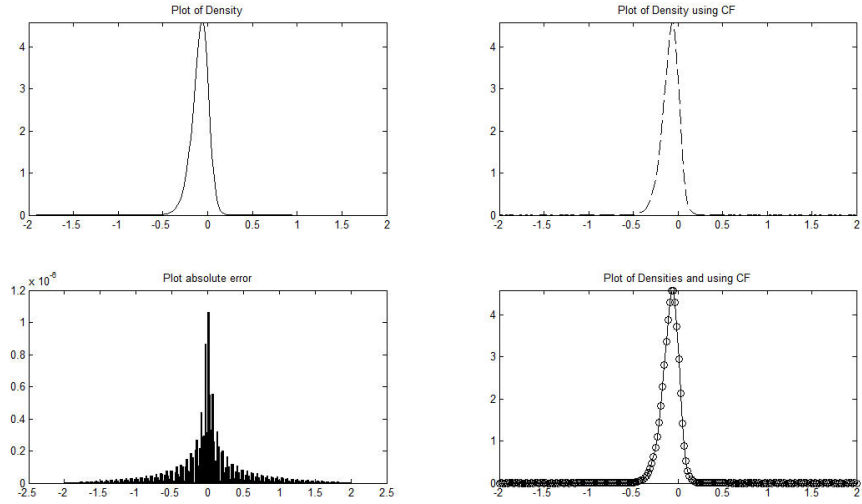


Figure 2: $T = 0.5$

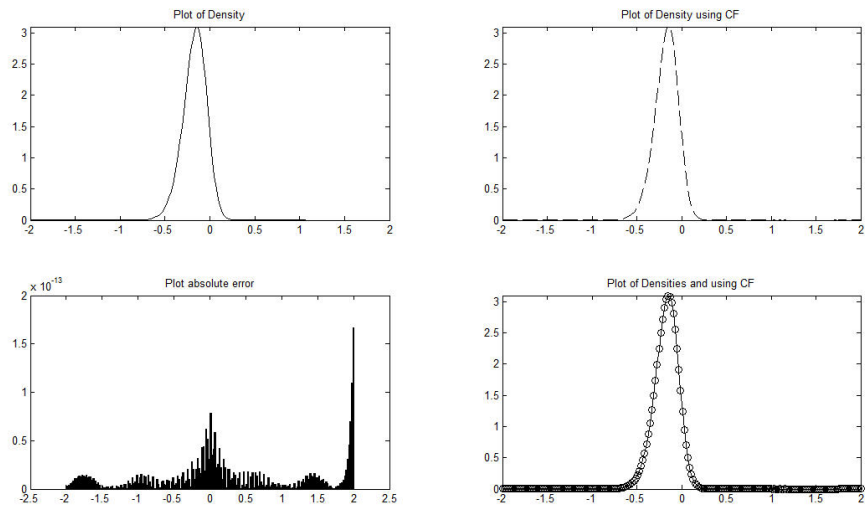


Figure 3: $T = 1.0$

3.5 Applying the Method to the VGOU model

For the Variance Gamma model with Gamma Ornstein-Uhlenbeck stochastic clock a closed form of the transition density is not known but the characteristic

function is given in closed form. We apply the inversion technique discussed before to this model. Again, we consider the following parameter sets for our numerical test:

Set	C	G	M	λ	a	b
Set 1	6.161	9.6443	16.026	1.679	0.3484	0.7664
Set 2	8.88	24.95	48.19	3.3	0.715	1.031
Set 3	6.47	11.10	33.41	0.94	0.63	1.47

For each of the parameter sets we plot the density obtained by using the characteristic function.

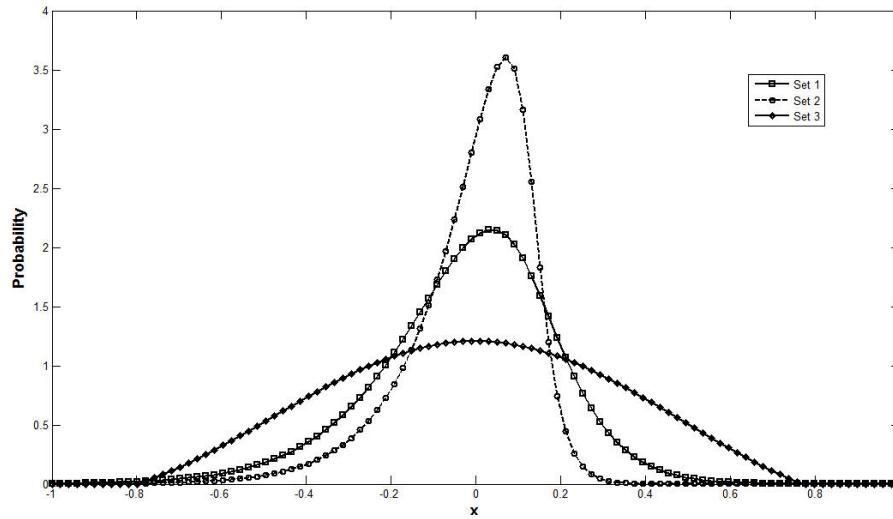


Figure 4: The probability density functions obtained using the characteristic function. We have indicated the sampled points. The lines are interpolated values

3.5.1 Varying a

We consider Set2 and show the effect of changing the parameter a. Figure 5 shows the effect to the distribution.

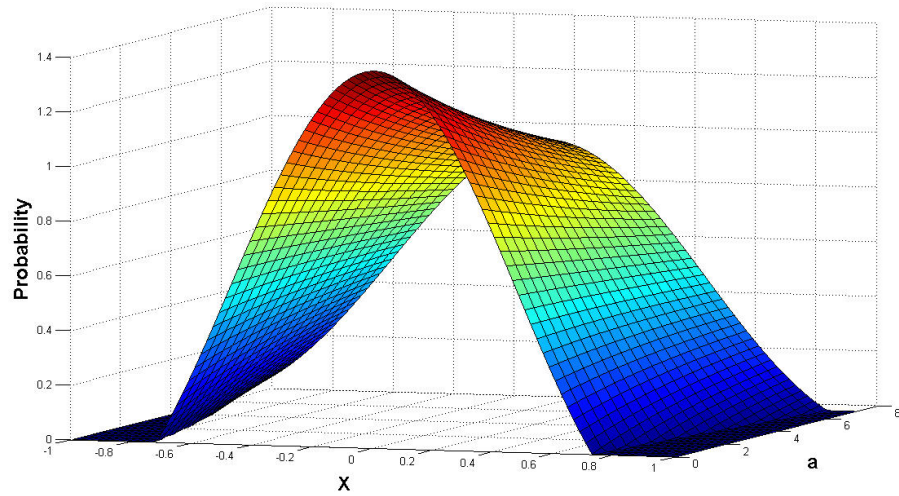


Figure 5: Changing the parameter a from Set2

3.5.2 Varying b

We consider Set2 and show the effect of changing the parameter b. Figure 6 shows the effect to the distribution.

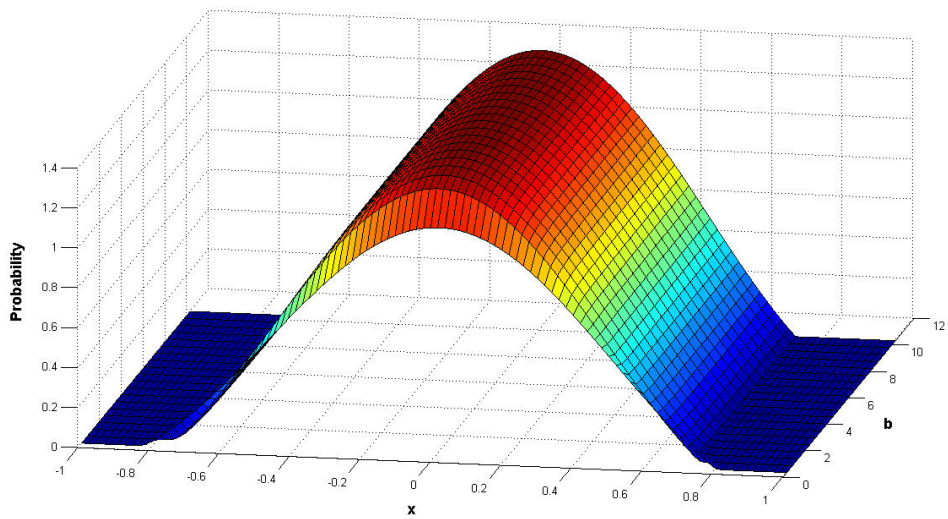


Figure 6: Changing the parameter a from Set2

4 Estimating Sensitivities using Proxy Methods

Recently, Fries and Kampen (2005), Fries and Joshi (2006) and Fries (2007) have applied the proxy simulation scheme to an Itô stochastic differential equation, especially to computing Greeks for TARNs in the Libor market model, and Kienitz (2008) has applied it to jump-diffusions and other Lévy processes based models.

We review briefly the method to obtain stable Greeks for discontinuous path-dependent payoff functions.

We take a stochastic process

$$(21) \quad S : t \mapsto S(t)$$

and consider a discretisation, $\tau = \{0 = t_0, t_1, \dots, t_{NTime} = T\}$, of the stochastic process $S(t)$ in the interval $[0, T]$ and take the time discrete processes, $S_*(t)$, $t \in \tau$ as an approximation to S .

$$(22) \quad S_* : t \mapsto S_*(t).$$

If an exact solution is available we use the time discrete process $S_*(t) = S(t)$, $t \in \tau$, directly. Furthermore, we take another approximation given by:

$$(23) \quad S_0 : t \mapsto S_0(t).$$

The latter can be an easy to implement scheme like an Euler scheme for example. We call (22) the target scheme and (23) the proxy scheme. For the sequel we need the transition kernel of (22) and (23). In practice, however, there are models where the transition kernel for the process (21) is only available through its characteristic function.

4.0.3 The Proxy Method and Algorithm

To derive approximations to the Greeks of an option with payoff h we have to approximate:

$$(24) \quad \frac{\partial}{\partial \Phi} \mathbb{E} [h(S^\Phi)] = \frac{\partial}{\partial \Phi} \int_{\mathbb{D}} h(x) \pi^\Phi(x) dx.$$

The domain on which the density is defined is denoted by \mathbb{D} , for example it is often $\mathbb{D} = \mathbb{R}$ or \mathbb{R}^+ . Now, we apply the integration by parts rule and get

$$\begin{aligned}
& \frac{\partial}{\partial \Phi} \mathbb{E} [h(S^\Phi)] \\
(25) \quad & \approx \frac{1}{2\epsilon} (\mathbb{E} [h(S_*^{\Phi+\epsilon})] - \mathbb{E} [h(S_*^{\Phi-\epsilon})]) \\
(26) \quad & = \int_{\mathbb{D}} h(y) \frac{1}{2\epsilon} (f_{S_*^{\Phi+\epsilon}}(y) - f_{S_*^{\Phi-\epsilon}}(y)) dy \\
(27) \quad & = \int_{\mathbb{D}} h(y) \frac{\frac{1}{2\epsilon} (f_{S_*^{\Phi+\epsilon}}(y) - f_{S_*^{\Phi-\epsilon}}(y))}{f_{S_0}(y)} f_{S_0}(y) dy \\
(28) \quad & \stackrel{\text{MC}}{\approx} \frac{1}{N} \sum_{i=1}^N h(S_{0,i}) \frac{1}{2\epsilon} (\omega^{\Phi+\epsilon}(S_{0,i}) - \omega^{\Phi-\epsilon}(S_{0,i}))
\end{aligned}$$

In the last step we used the notation:

$$(29) \quad \omega^{\Phi+\epsilon}(y) = \frac{f_{S_*^{\Phi+\epsilon}}(y)}{f_{S_0}(y)}.$$

We give the algorithm which implements the full proxy simulation. To this end we denote the number of simulations by $N\text{Sim}$, the number of time steps per simulation by $N\text{Time}$. The algorithm to compute the value of the Greek with respect to the disturbed initial parameter $\Phi \pm \epsilon$ is given by:

- For each run $i, i = 1, \dots, N\text{Sim}$
Simulate \hat{S}_j^i at $t_j \in \{t_1, \dots, t_{N\text{Time}} = T\}$
- Compute the weights $\omega_{i,j}^{\text{path}}$
- Compute the weights $\omega_i = \prod_{j=1}^{N\text{Time}} \omega_{i,j}^{\text{path}}$
- Compute the payoff

$$(30) \quad \hat{V} = \frac{1}{N\text{Sim}} \sum_{i=1}^{N\text{Sim}} h(\hat{S}_i) \omega_i(\hat{S}_i)$$

We used the fact that by equation (29) we can compute the weights $\omega_i, i = 1, \dots, N\text{Sim}$ for the i -th path by:

$$(31) \quad \omega_i(y) = \prod_{j=1}^{N\text{Time}} \frac{f^{\Phi+\epsilon}(y) - f^{\Phi-\epsilon}(y)}{f^\Phi(y)} = \prod_{j=1}^{N\text{Time}} \omega^{\Phi+\epsilon} - \omega^{\Phi-\epsilon}$$

Parameter	Proxy	Proxy Char
Set2	(Digital Option)	
Δ_A	0.02862	
Δ	0.02824 (5.116e-004)	0.02824 (5.116e-004)
Γ_A	-0.001128	
Γ	-0.001150(7.368e-005)	-0.001150(7.368e-005)
Vega _A	-0.900909100578273	
Vega	-0.9031(0.0710)	-0.9031(0.0710)

In our approach using the characteristic function we replace the densities $f^{\Phi+\epsilon}$, $f^{\Phi-\epsilon}$ and f^Φ by the numerically computed ones as described in section 3.2.

The mixed and the localised proxy methods are two variants of the full proxy scheme which can be applied to more effectively implement the algorithm. Since Fries and Kampen (2005), Fries and Joshi (2006) and Fries (2007) have described them in detail and we explored the applicability of them in Kienitz (2008) we do not cover the methods here.

5 Numerical Examples

For the asset price dynamics 1 we choose the following parameters:

$$S = 100, \quad r = 0.1 \ (r = 0.02), \quad d = 0$$

The maturity of the option is $T = 1$. We use $\Delta = 1$ and $\Delta = 1/12$ for the cases of a digital and respectively for the knock-out option.

5.1 The VG model

The Variance Gamma model can be seen as a benchmark. We applied the proxy method to derive sensitivities in Kienitz (2008) and obtained stable results. We are now going to compare the results using the approximation to the density using the characteristic function.

5.1.1 Digital Option

The standard proxy methods have shown to produce reliable and stable results when the payoff is discontinuous. We have chosen the case of a digital option as one representative because closed form solution exist to benchmark the Monte Carlo simulation method for the Variance Gamma model.

We consider a Digital option with strike price $K = 101$. The Vega is the change in the parameter σ and not with respect to implied volatility. We observe that there is virtually no difference between the method using the closed form expression and the estimator using the characteristic function. This is illustrated by figures 7 and 8 which give the running average of the estimators using the transition density in closed form and the density obtained from inverting the characteristic function. The latter samples are indicated by circles.

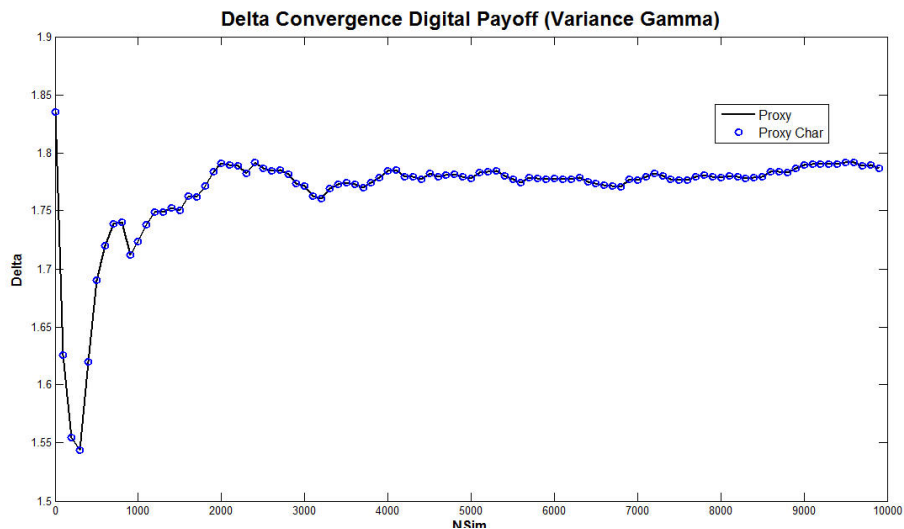


Figure 7: Proxy Simulation for a digital option using closed form and characteristic function version of the transition density for simulated Δ .

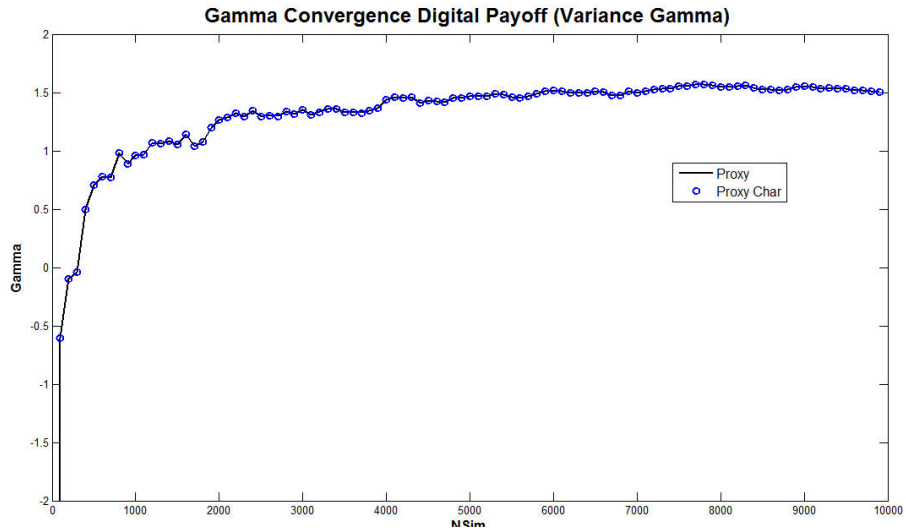


Figure 8: Proxy Simulation for a digital option using closed form and characteristic function version of the transition density for simulated Γ .

Set2	(Knock-Out option)	
Δ_{FD}	-0.2408	
Δ	-0.1464 (0.0279)	-0.1456 (0.0280)
Γ_{FD}	-1.0578	
Γ	0.1619(0.1891)	0.2328(0.1351)
Vega $_{FD}$		
Vega	35.2263(0.7343)	35.2263 (0.7343)

5.1.2 Knock-Out Options

For path-dependent options with discontinuous payoffs the proxy methods produces reliable results. We consider knock-out options and show that the accuracy heavily depends on the choice of parameters for A and M used to numerically invert the characteristic function.

For the knock-out option we consider a time-dependent barrier structure given by the lower bound $L(t)$ and the upper bound $U(t)$

$$L(t) = \begin{cases} 90 & \text{if } t \in (0, 0.5] \\ 85 & \text{if } t \in (0.5, 0.75] \\ 80 & \text{if } t \in (0.75, 1] \end{cases}$$

$$U(t) = \begin{cases} 110 & \text{if } t \in (0, 0.5] \\ 115 & \text{if } t \in (0.5, 0.75] \\ 120 & \text{if } t \in (0.75, 1] \end{cases}$$

We find the following results: The figures 9 and 10 show the proxy simulation using on the one hand side the closed form transition kernel and on the other hand side the transition density obtained using the characteristic function. We show how the estimates vary choosing the parameters A and M . The figures 11 and 12 illustrate the results.

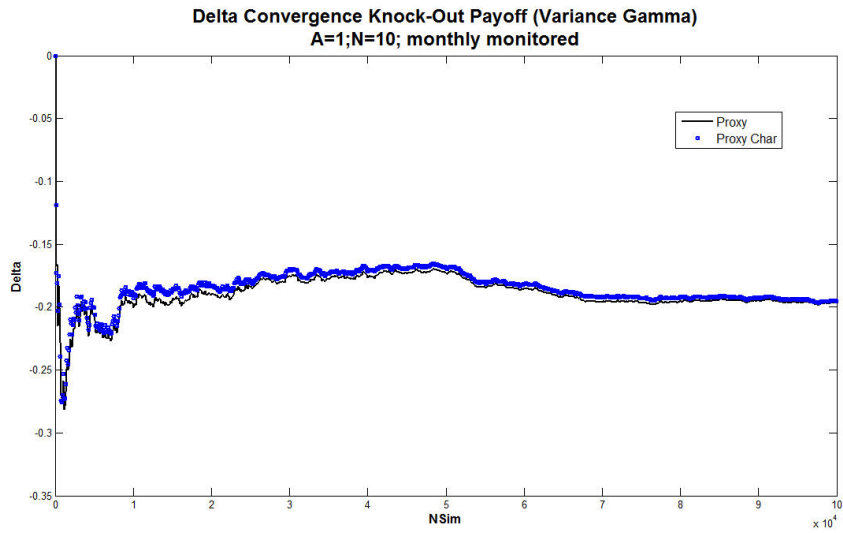


Figure 9: Proxy Simulation for a digital option using closed form and characteristic function version of the transition density for simulating Δ with $A = 1$ and $M = 10$

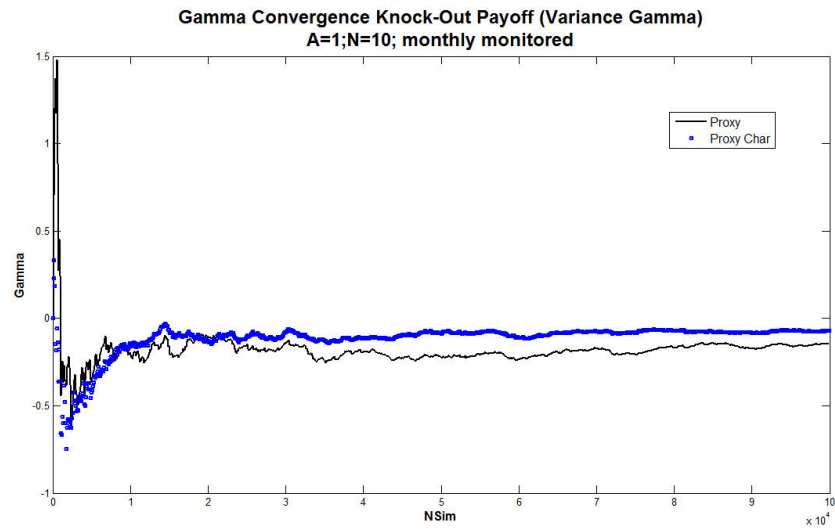


Figure 10: Proxy Simulation for a digital option using closed form and characteristic function version of the transition density for simulating Γ with $A = 1$ and $M = 10$

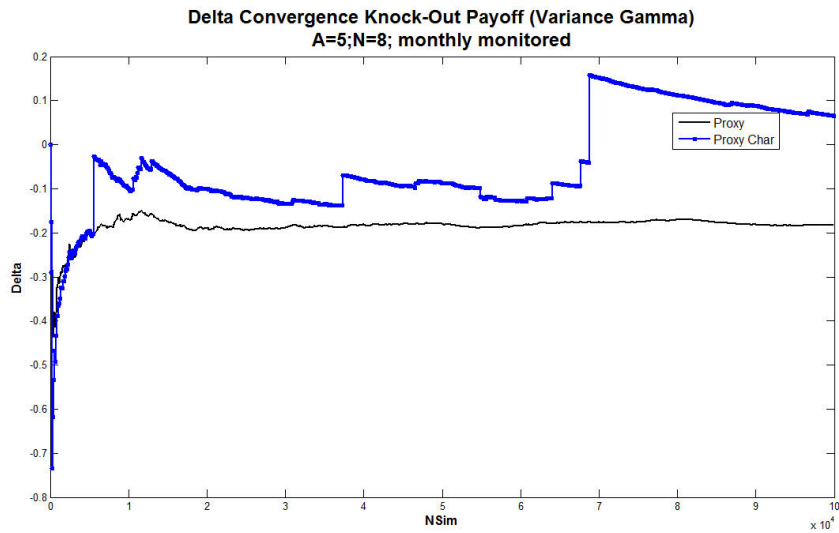


Figure 11: Proxy Simulation for a digital option using closed form and characteristic function version of the transition density for simulating Δ with $A = 5$ and $M = 8$

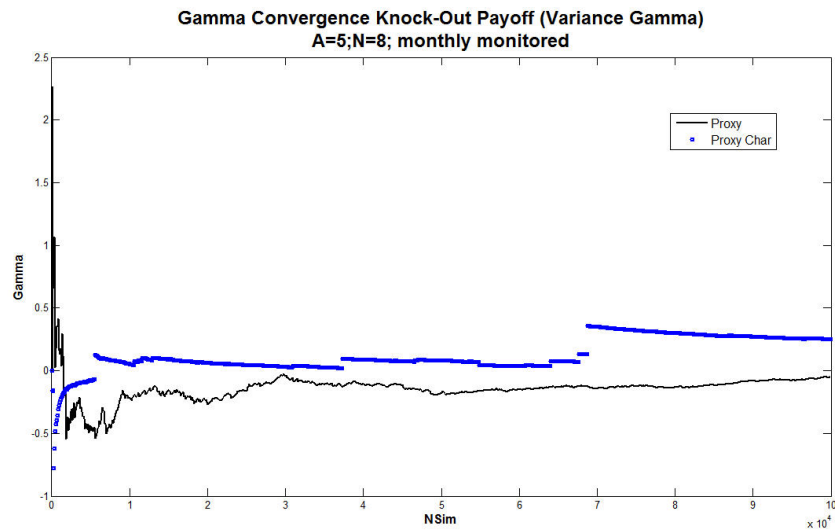


Figure 12: Proxy Simulation for a digital option using closed form and characteristic function version of the transition density for simulating Γ with $A = 5$ and $M = 8$

5.2 The VGOU model

We consider the VGOU model as introduced in 2.3. In this case we take the same parameters as before but $r = 0.02$.

5.2.1 Digital Option

The standard proxy methods have shown to produce reliable and stable results when the payoff is discontinuous. We have chosen the case of a digital option as one representative because closed form solution exist to benchmark the Monte Carlo simulation method for the Variance Gamma model.

We again consider a Digital option with strike price $K = 101$.

Parameter	Proxy	Proxy Char
Set2	(Digital Option)	
Δ_A	0.0275	
Δ	0.0283 (2.4050e-004)	0.0250 (0.0490)
Γ_A	-0.0017	
Γ	-0.0018(4.8989e-005)	0.0980(4.6930)

This is furthermore illustrated by the figures 13 and 14 which show the running averages of the estimator using the proxy method with transition kernels obtained by inverting the characteristic function and the estimator using the finite difference method.

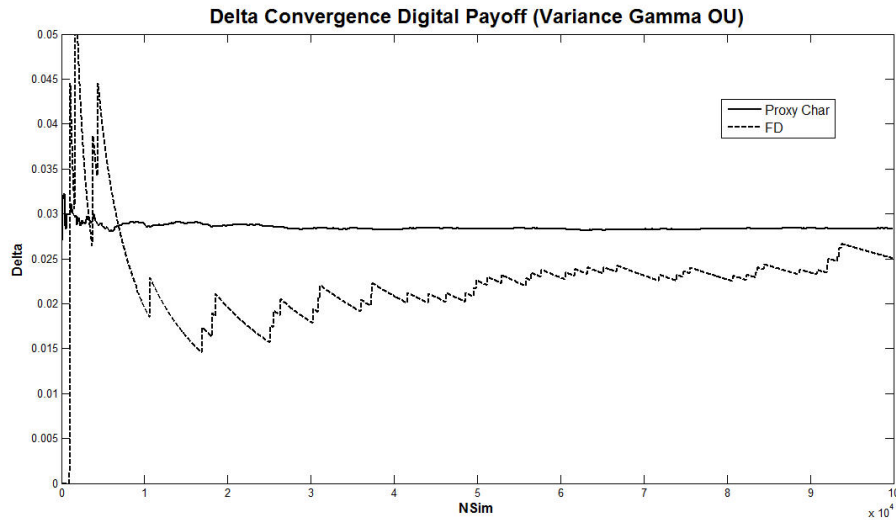


Figure 13: Proxy Simulation for a digital option

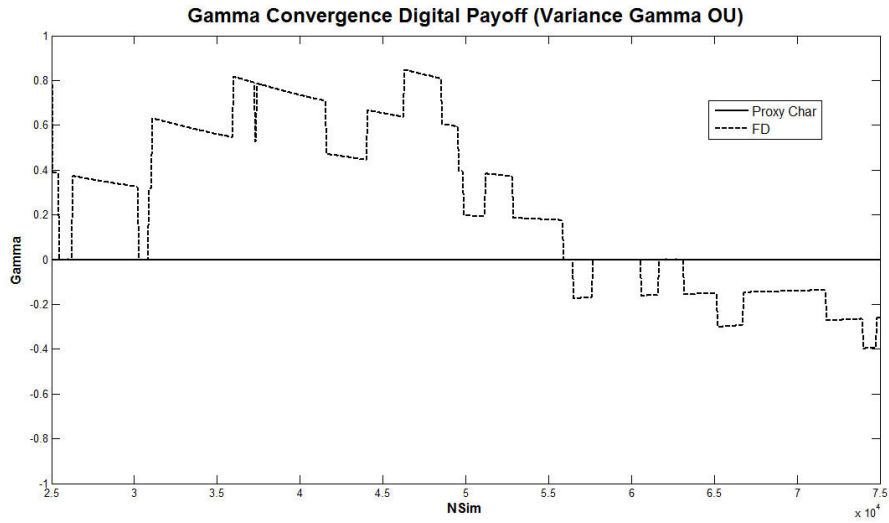


Figure 14: Proxy Simulation for a digital option

5.2.2 Knock-Out Option

We again consider the knock-out option from the previous section. The table summarizes the results.

Set2	(Knock-Out Option)	
Parameter	5 Steps per month	10 Steps per month
Δ_{FD}	0.1084	0.1417
Δ	-0.0734(0.0041)	-0.0688 (0.0041)
Γ_{FD}	-3.6413	2.6507
Γ	0.2100(0.0092)	0.2055(0.0098)

Figures 15, 17, 16 and 18 show the running average of the estimators using the numerical inversion to get the transition density and the finite difference estimators. We considered 10 and 20 integration steps between the monitored times.

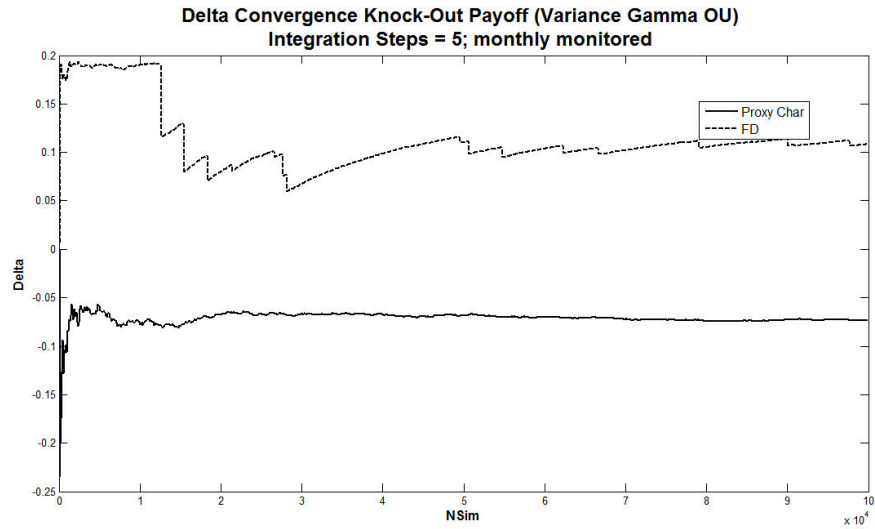


Figure 15: Proxy Simulation Delta of a knock-out option using 10 steps to determining the stochastic clock

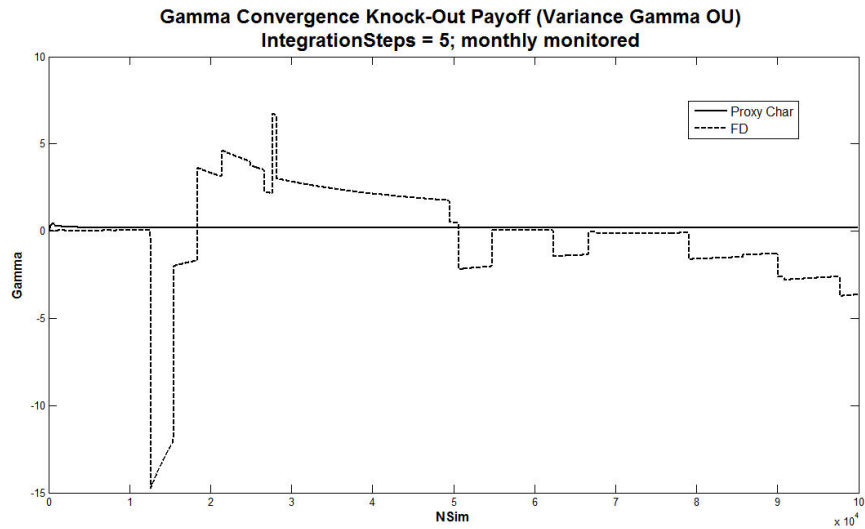


Figure 16: Proxy Simulation for Gamma of a knock-out option using 10 steps to determine the stochastic clock

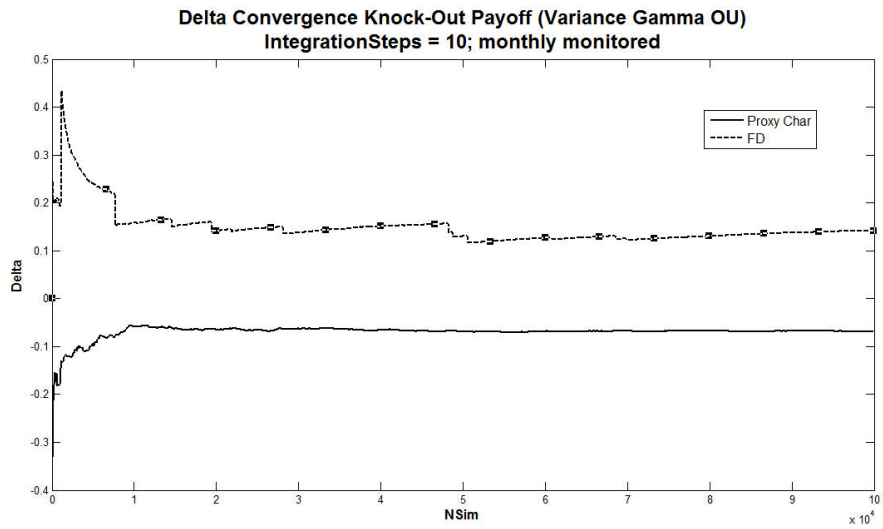


Figure 17: Proxy Simulation for Delta of a knock-out option using 20 steps to determine the stochastic clock

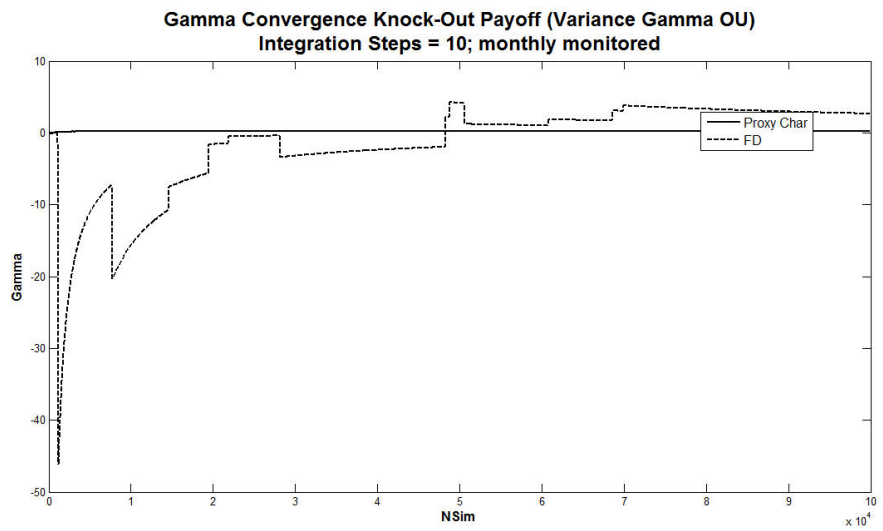


Figure 18: Proxy Simulation for Gamma of a knock-out option using 20 steps to determine the stochastic clock

6 Summary and Conclusion

We have reviewed how to approximate the probability density given the characteristic function. We applied the numerical inversion to the transition probabilities of the VG and VGOU models. Then, we applied the techniques to the estimation of sensitivities for options with path-dependent, discontinuous payoff functions. We observed that the method only using the characteristic function lead to reasonable and stable results when choosing appropriate parameters. The dependence on the parameters has been illustrated. We are now in a position to apply sophisticated financial models based stochastic volatility Lévy models for pricing and risk management.

Finally, we have to mention that the methods methods are time consuming since at each step a numerical inversion has to be done to obtain the value of the transition density. Further research has to be done to reduce the execution time by using more efficient numerical inversion methods and applying variance reduction techniques like stratified sampling or bridge techniques.

We thank Philipp Beyer and Daniel Wetterau for fruitful discussions.

References

- P. Beyer and J. Kienitz. Pricing Forward Start Option in Models based on Lévy Processes. *forthcoming, www.SSRN.com*, 2009a.
- P. Beyer and J. Kienitz. Forward Implied Volatility Surfaces, Calibration, Pricing and Risk Management for Financial Models based on Lévy Processes. *forthcoming, www.SSRN.com*, 2009b.
- Glasserman P. Chen, N. Malliavin Greeks without Malliavin Calculus. *Preprint*, 2006.
- R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall, 2004.
- Johannson M.P. Davis, M.H.A. Malliavin monte carlo greeks for jump diffusions. *Preprint*, 2004.
- D. Duffy and J. Kienitz. *Monte Carlo Frameworks - Building Customisable and High-performance C++ Applications*. John Wiley & Sons, Chichester, 2009.
- Privault N. El-Khatib, Y. Computations of Greeks in a market with jumps via the Malliavin calculus. *Finance and Stochastics*, 8:161–179, 2004.
- F. Fang and K. Osterlee. A Novel Pricing Method for European Option based on Fourier-Cosine Series Expansions. *Munich Personal RePEc ARchive*, 9319, 2008.
- C. Fries. Localized Proxy Simulation Schemes for Generic and Robust Monte-Carlo Greeks. *www.chrsitian-fries.de*, 2007.

- C. Fries and M. Joshi. Partial Proxy Simulation Schemes for Generic and Robust Monte-Carlo Greeks. *www.christian-fries.de*, 2006.
- C. Fries and J. Kampen. Proxy Simulation Schmes for generic and robust Monte-Carlo sensitivities and high accuracy drift approximation (with applications to the Libor Market Model. *preprint*, 2005.
- P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer, Berlin, Heidelberg, New York, 2004.
- P. Glasserman and Z. Liu. Sensitivity Estimates From Characteristic Functions. 2008.
- P. Hughett. Error Bounds for Numerical Inversion of a Probability Characteristic Function. *SIAM Journal of Numerical Analysis*, 40(3):1368–1392, 1998.
- P. Jäckel. *Monte Carlo Methods in Finance*. Wiley, 2002.
- J. Kienitz. Stochastic Processes in Finance I. *Wilmott magazine*, 2007a.
- J. Kienitz. Stochastic Processes in Finance II. *Wilmott magazine*, 6, 2007b.
- J. Kienitz. Stochastic Processes in Finance III. *Wilmott magazine*, forthcoming, 2009.
- J. Kienitz. A Note on Monte Carlo Greeks for Jump Diffusions and other Lévy Models. 2008.
- Solé J.L. Utzet F. Vives J. León, J.a. On Lévy processes, Malliavin calculus and market models with jumps. *Finance and Stochastics*, 6:197–225, 2002.
- Carr P.P. Chang E.C. Madan, D.B. The variance gamma process and option pricing. *European Finance Review*, 2:79–105, 1998.
- W. Schoutens, E. Simons, and J. Tistaert. A Perfect Calibration! Now What? *Wilmott Magazine*, 2, 2004.